Selected problems & solutions to Royden.

1 Chapter 2

Problem (1)**.** Prove that if A and B are two sets in $\mathcal{A}$ with $A \subseteq B$, then $m(A) \leq m(B)$. This property is called monotonicity.

Proof. If $m(B) = \infty$, then clearly the inequality holds true. Assume $m(B) < \infty$ and write $B = A \cup (B \setminus A)$, where $A \cap (B \setminus A) = \emptyset$. Then, by σ-additivity,

$$m(B) = m(A \cup (B \setminus A)) = m(A) + m(B \setminus A)$$

thus

$$m(B) - m(A) = m(B \setminus A) \geq 0$$

which implies that

$$m(B) \geq m(A).$$

□

Problem (2)**.** Prove that if there is a set A in the collection $\mathcal{A}$ for which $m(A) < \infty$, then $m(\emptyset) = 0$.

Proof. Observe that $A = \emptyset \cup A$, where $\emptyset \cap A = \emptyset$. Thus, by σ-additivity,

$$m(A) = m(A \cup \emptyset) = m(A) + m(\emptyset)$$

and since $m(A)$ is finite, we can subtract to conclude that

$$0 = m(\emptyset).$$

□

Problem (3)**.** Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$. Prove that $m\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m(E_k)$.

Proof. Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$, and define $\{B_k\}_{k=1}^{\infty}$ as follows:

$$\begin{aligned} B_1 &= E_1 \\ B_2 &= E_2 \setminus B_1 \\ &\vdots \\ B_n &= E_n \setminus \left(\bigcup_{j=1}^{n-1} E_j \right) \\ &\vdots \end{aligned}$$

Clearly, $\bigcup_{k=1}^{\infty} B_k \subseteq \bigcup_{k=1}^{\infty} E_k$.

On the other hand, let $x \in \bigcup_{k=1}^{\infty} E_k$ and let k_0 be the first $k \in \mathbb{Z}$ such that $x \in E_{k_0}$. Then, $x \in B_{k_0}$, and thus $x \in \bigcup_{k=1}^{\infty} B_k$. Therefore, $\bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} B_k$. Then, by σ-additivity,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = m\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} m(B_k)$$

and by monotonicity,

$$\sum_{k=1}^{\infty} m(B_k) \leq \sum_{k=1}^{\infty} m(E_k)$$

. □

Problem (4)**.** The set function, c, defined on all subsets of $\mathbb{R}$ is defined as follows:

$$c(E) = \begin{cases} \infty, & E \text{ is finite} \\ |E|, & E \text{ is finite} \\ 0, & E = \emptyset \end{cases}$$

Show that c is countably additive and translation invariant.

Proof. (Alexander's proof)
First we will show that c is countably additive over countable disjoint unions of sets. Let $\{E_k\}_{k=1}^{\infty}$ be a countably infinite collection of disjoint subsets of $\mathbb{R}$. If E_i is infinite for some i, then $\bigcup_{k=1}^{\infty} E_k$ is an infinite set, and

$$c\left(\bigcup_{k=1}^{\infty} E_k\right) = \infty = \sum_{k=1}^{\infty} c(E_k).$$

Now assume that E_i is not infinite for any i. Since the subsets are disjoint,

$$\sum_{k=1}^{\infty} c(E_k) = \sum_{k=1}^{\infty} |E_k| = \left|\bigcup_{k=1}^{\infty} E_k\right| = c\left(\bigcup_{k=1}^{\infty} E_k\right)$$

and thus c is countably additive.

Next, we will show that c is translation invariant. First, consider the case where E is infinite. Then $c(E) = \infty$ and $c(E+y) = \infty$. If E is empty, then $E+y$ is empty, and $c(E) = 0 = c(E+y)$. If E is finite and nonempty, then we can say $|E| = n$, and $E = \{e_1, e_2, \ldots, e_n\}$ for $e_i \in \mathbb{R}$. Thus, $E + y = \{e_1 + y, e_2 + y, \ldots, e_n + y\}$ and $|E + y| = y$. Thus, $c(E) = c(E + y)$, and c is translation invariant. □

Problem (5)**.** Prove the interval $[0, 1]$ is uncountable.

Proof. Assume to the contrary $[0, 1]$ is countable. Then, since the measure of a countable set is zero, $m^*([0, 1]) = 0$. However, $m^*([0, 1]) = \ell([0, 1]) = 1 - 0 = 1$, a contradiction, since $1 \neq 0$. Thus, $[0, 1]$ is not countable. □

Problem (6)**.** Let A be the set of irrational numbers in $[0, 1]$. Prove that $m^*(A) = 1$.

Proof. Observe that $[0, 1] = ([0, 1] \cap A) \cup ([0, 1] \cap \mathbb{Q}) = A \cup ([0, 1] \cap \mathbb{Q})$. Since $A \cap ([0, 1] \cap \mathbb{Q}) = \emptyset$, by σ-additivity,

$$\begin{aligned} m^*([0, 1]) &= m^*(A \cup ([0, 1] \cap \mathbb{Q})) \\ &= m^*(A) + m^*([0, 1] \cap \mathbb{Q}) \\ &= m^*(A) + 0 \end{aligned}$$

and thus we see that $m^*(A) = m^*([0, 1]) = 1$. □

Problem (7)**.** A set of real numbers is said to be a G_δ set provided it is the intersection of a countable collection of open sets. Show that for any bounded set E, there is a G_δ set G such that

$$E \subseteq G \text{ and } m^*(G) = m^*(E).$$

Proof. By the definition of Lebesgue outer measure,

$$m^*(E) = \text{glb}\left\{\sum_{k=1}^{\infty} \ell(I_k) : E \subseteq \bigcup_{k=1}^{\infty} I_k \text{ and } I_k \text{ open, nonempty}\right\}.$$

By definition of glb, there is an open cover of E, $\{I_{j,n}\}_{j=1}^{\infty}$, for each $\frac{1}{n}$, where

$$\sum_{k=1}^{\infty} \ell(I_{k,n}) \leq m^*(E) + \frac{1}{n}.$$

Let $\mathcal{O}_n = \bigcup_{j=1}^{\infty} I_{j,n}$, where $\mathcal{O}_n$ is an open set covering E for each n. Then $E \subseteq \bigcap_{n=1}^{\infty} \mathcal{O}_n$. Let $G = \bigcap_{n=1}^{\infty} \mathcal{O}_n$. Note that G is a G_δ set and $E \subseteq G$.

By monotonicity, $m^*(E) \leq m^*(G)$. Also, since $\mathcal{O}_n \supseteq G$, by monotonicity,

$$m^*(E) \leq m^*(G) \leq m^*(\mathcal{O}_n).$$

Note that one covering of $\mathcal{O}_n$ is is $\{I_{k,n}\}_{k=1}^{\infty}$, ie, $\mathcal{O}_n$ covers itself. Since $m^*(\mathcal{O}_n)$ is the glb of all open covers of $\mathcal{O}_n$,

$$m^*(\mathcal{O}_n) \leq \sum_{j=1}^{\infty} \ell(I_{j,n}) \leq m^*(E) + \frac{1}{n}.$$

Thus, $m^*(E) \leq m^*(G) \leq m^*(E) + \frac{1}{n}$. Thus, $m^*(E) = m^*(G)$. □

Problem (8)**.** Let B be the set of rational numbers in $[0, 1]$ and let $\{I_k\}_{k=1}^{\infty}$ be a finite collection of open intervals that covers B. Prove that $\sum_{k=1}^{\infty} m^*(I_k) \geq 1$.

Proof. (Alexander's proof)
Since $\{I_k\}_{k=1}^{\infty}$ is a countable collection of open intervals, we can write out the intervals ordered by left endpoint, i.e., write them as $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$, where $a_1 \leq a_2 \leq \cdots \leq a_n$. Note that $a_j \leq b_{j-1}$ for all $1 \leq j \leq n$, because otherwise there is an interval $(b_{j-1}, a_j) \not\subseteq \bigcup_{k=1}^{\infty} I_k$, a contradiction (since this interval contains rational numbers). We know that $0 \in (a_1, b_1)$ and $1 \in (a_n, b_n)$, and thus $a_1 \leq 0$ and $b_n \geq 1$. Thus,

$$\sum_{k=1}^{\infty} m^*(I_k) = \sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} \ell((a_k, b_k))$$

where

$$\begin{aligned}
\sum_{k=1}^{\infty} \ell((a_k, b_k)) &= (b_n - a_n) + (b_{n-1} - a_{n-1}) + \cdots + (b_1 - a_1) \\
&= b_n + (b_{n-1} - a_n) + (b_{n-2} - a_{n-1}) + \cdots + (b_1 - a_2) - a_1 \\
&\geq b_n - a_1 \\
&\geq 1
\end{aligned}$$

Thus, $\sum_{k=1}^{\infty} m^*(I_k) \geq 1$. □

Problem (9)**.** Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

Proof. If $m^*(B) = \infty$, then the inequality holds. Assume $m^*(B) < \infty$. Generally, $m^*(B) \leq m^*(A \cup B)$. Furthermore, by σ-additivity, we know that $m^*(A \cup B) \leq m^*(A) + m^*(B) = 0 + m^*(B)$. Thus $m^*(B) = m^*(A \cup B)$. □

Problem (10)**.** Let A and B be bounded sets for which there is an $\alpha > 0$ such that $|a - b| \geq \alpha$ for all $a \in A, b \in B$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.

Proof. Let $\{I_j\}_{j=1}^{\infty}$ be a covering of $A \cup B$ with open intervals $\ell(I_j) < \frac{\alpha}{2}$ and $\sum_{j=1}^{\infty} \ell(I_j) \leq m^*(A \cup B) + \epsilon$. Note that if $I_j \cap A \neq \emptyset$ then $I_j \cap B = \emptyset$. Thus, we can split the intervals into a covering of A and a covering of B. Say $\{I_j^A\}_{j=1}^{\infty}$ covers A, where $I_j^A \cap A \neq \emptyset$ for all j, and $\{I_j^B\}_{j=1}^{\infty}$ covers B, where $I_j^B \cap B \neq \emptyset$ for all j.

Now, observe the outer measure:

$$m^*(A \cup B) + \epsilon \geq \sum_{j=1}^{\infty} \ell(I_j) \geq \sum_{j=1}^{\infty} \ell(I_j^A) + \sum_{j=1}^{\infty} \ell(I_j^B) \geq m^*(A) + m^*(B).$$

Since by subadditivity we know that $m^*(A) + m^*(B) \geq m^*(A \cup B)$, then

$$m^*(A) + m^*(B) = m^*(A \cup B).$$

□

Problem (11)**.** Prove that if a σ-algebra of subsets of $\mathbb{R}$ contains intervals of the form (a, ∞), then it contains all intervals.

Proof. Let $\mathcal{A}$ be a σ-algebra which contains all real intervals of the form (a, ∞). Then, for $a < b$,

- $(-\infty, a] = (a, \infty)^C \in \mathcal{A}$
- $(-\infty, a) = \bigcup_{k=1}^{\infty} (-\infty, a - \frac{1}{k}] \in \mathcal{A}$
- $[a, \infty) = (-\infty, a)^C \in \mathcal{A}$
- $(-\infty, b] \cap [a, \infty) = [a, b] \in \mathcal{A}$
- $(-\infty, b) \cap (a, \infty) = (a, b) \in \mathcal{A}$
- $(-\infty, b) \cap [a, \infty) = [a, b) \in \mathcal{A}$
- $(-\infty, b] \cap (a, \infty) = (a, b] \in \mathcal{A}$
- $(-\infty, a] \cap [a, \infty) = \{a\} \in \mathcal{A}$
- $(-\infty, a] \cap [b, \infty) = \emptyset \in \mathcal{A}$
- $\bigcup_{k=1}^{\infty} (-k, k) = \mathbb{R} \in \mathcal{A}$

Thus, $\mathcal{A}$ contains all intervals. □

Problem (12)**.** Every interval is a Borel set.

Proof. The collection $\mathcal{B}$ of Borel sets of $\mathbb{R}$ is the smallest σ-algebra of sets of $\mathbb{R}$ containing all open sets in $\mathbb{R}$. Since every open set is in $\mathcal{B}$, then every set of real numbers of the form (a, ∞) is in $\mathcal{B}$. Thus, by the previous exercise, every interval is a Borel set. □

Problem (13)**.** Show that (i) the translate of an F_σ set is also F_σ, and (ii) the translate of a G_δ set is also G_δ, and (iii) the translate of a set of measure zero is also measure zero.

Proof. (i) Let F be an F_σ set. Then, F is a countable union of closed sets. Say $F = \cup_{k=1}^{\infty} F_k$, where F_k is closed for all k. Then,

$$F + y = (\cup_{k=1}^{\infty} F_k) + y = \cup_{k=1}^{\infty}(F_k + y)$$

where $F_k + y$ is closed for all k. Thus, $F + y$ is an F_σ set.

(ii) Let G be a G_δ set. Then, $G = \bigcap_{k=1}^{\infty} \mathcal{O}_k$ where $\mathcal{O}_k$ is open for all k. Then,

$$G + y = \left(\bigcap_{k=1}^{\infty} \mathcal{O}_k\right) + y = \bigcap_{k=1}^{\infty} (\mathcal{O}_k + y)$$

where $\mathcal{O}_k + y$ is open for all k. Thus, $G + y$ is G_δ.

(iii) Suppose E is a set of measure zero, ie, $m^*(E) = 0$. Let $\{I_j\}_{j=1}^{\infty}$ be an open covering of E with $\sum_{j=1}^{\infty} \ell(I_j) < \epsilon$ for each $\epsilon > 0$. Then,

$$E + y \subset \left(\bigcup_{j=1}^{\infty} I_j\right) + y = \bigcup_{j=1}^{\infty} (I_j + y).$$

Thus,

$$m^*(E + y) \leq \sum_{j=1}^{\infty} \ell(I_j + y) = \sum_{j=1}^{\infty} \ell(I_j) < \epsilon$$

and therefore $m^*(E + y) = 0$. □

Problem (14)**.** If E has positive outer measure, then there is a bounded subset of E that also has positive outer measure.

Proof. (Alexander's proof)
If E is bounded, then we are done. Assume E is not bounded. Then,

$$E = \bigcup_{n \in \mathbb{Z}} E \cap [n, n+1].$$

By σ-subadditivity,

$$0 < m^*(E) = m^*\left(\bigcup_{n \in \mathbb{Z}} E \cap [n, n+1]\right) \leq \sum_{n \in \mathbb{Z}} m^*(E \cap [n, n+1])$$

Thus, there is at least one $m \in \mathbb{Z}$ such that $m^*(E \cap [m, m+1]) > 0$. Therefore, $E \cap [m, m+1]$ is a bounded subset of E with positive outer measure. □

Problem (15)**.** Show that if E has finite measure and $\epsilon > 0$, then E is the disjoint union of a finite number of measurable sets, each of which has measure at most ϵ.

Proof. (Alexander's proof)
Assume a set A has finite outer measure, $m^*(A) < \infty$. Let $E_n = (-n, n] \cap A$. Note that

$$m^*(A) = \lim_{n \to \infty} m^*((-n, n] \cap A) = \lim_{n \to \infty} m^*(E_n)$$

and thus

$$|m^*(A) - m^*(E_n)| < \epsilon$$

which implies by the incision property that

$$m^*(A \setminus E_n) \leq |m^*(A) - m^*(E_n)| < \epsilon.$$

Thus, we can consider the set $E_n = E$, a bounded set of finite measure. Say that $\text{glb}(E) = x$ and $\text{lub}(E) = y$, and let n be the integer such that $y \in [x + (n-1)\epsilon, x + n\epsilon)$. Then, the (measurable) intervals

$$\begin{aligned} I_1 &= [x, x+\epsilon) \\ I_2 &= [x+\epsilon, x+2\epsilon) \\ &\vdots \\ I_{n-1} &= [x+(n-2)\epsilon, x+(n-1)\epsilon) \\ I_n &= [x+(n-1)\epsilon, x+n\epsilon) \end{aligned}$$

is a covering of E, where $I_j \cap I_k = \emptyset$ and $m^*(I_k) = \epsilon$ for all j, k. Now, define

$$\begin{aligned} J_1 &= E \cap I_1 \\ J_2 &= E \cap I_2 \\ &\vdots \\ J_n &= E \cap I_n \end{aligned}$$

Since $I_j \cap I_k = \emptyset$, then $J_j \cap J_k = \emptyset$ for all j, k. Furthermore, $0 \leq m^*(J_k) \leq \epsilon$, and

$$\bigcup_{k=1}^{n} J_k = \bigcup_{k=1}^{n} E \cap I_k = (E \cap I_1) \cup (E \cap I_2) \cup \cdots \cup (E \cap I_n) = E \cap (\cup_{k=1}^{n} I_k) = E,$$

since $E \subseteq \cup_{k=1}^{n} I_k$. Thus, we see that E is the disjoint union of a finite number of measurable sets, each of which has measure at most ϵ. □

Problem (16)**.** Complete the proof to Theorem 11 by showing that measurablility is equivalent to (iii) and also equilvalent to (iv).

(iii): For each $\epsilon > 0$, there is a closed set F contained in E for which $m^*(E \setminus F) < \epsilon$.

(iv): There is an F_σ set F contained in E for which $m^*(E \setminus F) = 0$.

Proof. (iii): Since E is measurable, then E^c is also measurable, and by Theorem 11 part (ii), there is a G_δ set G containing E^c for which $m^*(G \setminus E^c) = 0 < \epsilon$. This implies that $F = G^c \subset E$ and F is clsoed. Furthermore, $E \setminus F = E \cap G = G \cap E = G \setminus E^c$. Therefore, $m^*(E \setminus F) = m^*(G \setminus E^c) < \epsilon$.

(iv): Given any $\epsilon > 0$, there is an $F_n \subset E$ such that $m^*(E \setminus F_n) < \epsilon$ by party (iii). Set $F_\sigma = \bigcup_{n=1}^{\infty} F_n$. Then,

$$m^*(E \setminus F_\sigma) = m^*\left(E \setminus \left(\bigcup_{n=1}^{\infty} F_n\right)\right) \leq m^*(E \setminus F_n) < \epsilon.$$

Thus, since ϵ can be arbitrarily small, $m^*(E \setminus F_\sigma) = 0$.

Lastly, we want to show that (iv) implies measurablility. So, assume there is an F_σ set F contained in E for which $m^*(E \setminus F) = 0$. Note that $E = F \cup (E \setminus F)$. We know that $E \setminus F$ is measurable since any set with outer measure zero is measurable, and F is F_σ and thus measurable. Because countable unions of measurable sets is measurable, the entire right hand side of the equation is measurable. Thus, E is measurable. □

Problem (17)**.** Show that a set E is measurable if and only if for each $\epsilon > 0$, there is a closed set F and open set $\mathcal{O}$ for which $F \subseteq \mathcal{O} \subseteq E$ and $m^*(\mathcal{O} \setminus F) < \epsilon$.

Proof. " $\Rightarrow$ " Assume E is measurable. Then by Theorem 11, there is a closed set $F \subseteq E$ such that $m^*(E \setminus F) < \frac{\epsilon}{2}$ and an open set $\mathcal{O} \supseteq E$ such that $m^*(\mathcal{O} \setminus E) < \frac{\epsilon}{2}$. Furthermore,

$$m^*(\mathcal{O} \setminus F) = m^*((E \setminus F) \cup (\mathcal{O} \setminus E)) = m^*(E \setminus F) + m^*(\mathcal{O} \setminus E) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

" $\Leftarrow$ " Conversely, assume $m^*(\mathcal{O} \setminus F) < \epsilon$. Since $E \setminus F \subset \mathcal{O} \setminus F$, mononicity implies that $m^*(E \setminus F) < m^*(\mathcal{O} \setminus F) < \epsilon$. Thus, by Theorem 11, E is measurable. □

Problem (18)**.** Let $m(E) < \infty$. Show that there is an F_σ set F and G_δ set G such that $F \subseteq E \subseteq G$ and $m^*(F) = m^*(E) = m^*(G)$.

Proof. Assume E has finite measure. Then, by Theorem 11, there is an F_σ set F such that $F \subseteq E$ and $m^*(E \setminus F) = 0$ and a G_δ set G such that $E \subseteq G$ and $m^*(G \setminus E) = 0$.

Since E is measurable, we can apply the excision property to conclude that

$$m^*(E \setminus F) = m^*(E) - m^*(F) = 0 \text{ and } m^*(G \setminus E) = m^*(G) - m^*(E) = 0$$

and thus

$$m^*(E) = m^*(F) \text{ and } m^*(G) = m^*(E).$$

□

Problem (19*)**.** Assume E has finite outer measure. Show that if E is not measurable, then there is an open set $\mathcal{O}$ containing E that has finite outer measure and for which

$$m^*(\mathcal{O} \setminus E) > m^*(\mathcal{O}) - m^*(E).$$

Proof. Assume E is not measurable and assume to the contrary that for all open sets $\mathcal{O}$ containing E, $m^*(\mathcal{O} \setminus E) \leq m^*(\mathcal{O}) - m^*(E)$. By the definition of outer measure, there is a collection of intervals $\{I_j\}_{j=1}^\infty$ such that $\sum_{j=1}^\infty \ell(I_j) \leq m^*(E) + \epsilon$.

Consider $\mathcal{O} = \bigcup_{j=1}^\infty I_j$. Then,

$$m^*(\mathcal{O}) \leq \sum_{j=1}^\infty \ell(I_j) \leq m^*(E) + \epsilon.$$

Then, $m^*(\mathcal{O}) - m^*(E) < \epsilon$, which implies by our contradictory assumption that $m^*(\mathcal{O} \setminus E) < \epsilon$. Since ϵ is arbitary, this implies that E is measurable by Theorem 11, a contradiction. □

Problem (20)**.** Let E have finite outer measure. Show that E is measurable if and only if for each open, bounded interval (a, b),

$$b - a = m^*((a, b) \cap E) + m^*((a, b) \setminus E)$$

Proof. " $\Rightarrow$ " Assume E is measurable and consider the open, bounded interval (a, b). Then,

$$b - a = m^*((a,b)) = m^*((a,b) \cap E) + m^*((a,b) \setminus E)).$$

" $\Leftarrow$ " Converesly, let $b - a = m^*((a,b) \cap E) + m^*((a,b) \setminus E)$. Note that

$$m^*(E) = \text{glb}\left\{\sum_{k=1}^{\infty} \ell(I_k) : \bigcup_{k=1}^{\infty} I_k \supseteq E\right\}.$$

Let $\epsilon > 0$. There exists a collection of intervals $\{I_k\}_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} \ell(I_k) \leq m^*(E) + \epsilon$. Denote each $I_k = (a_k, b_k)$, and let $\mathcal{O} = \bigcup_{k=1}^{\infty}(a_k, b_k)$. Then,

$$\sum_{k=1}^{\infty} \ell((a_k, b_k)) = \sum_{k=1}^{\infty}[m^*((a_k, b_k) \cap E) + m^*((a_k, b_k) \cap E^c)] \leq m^*(E) + \epsilon.$$

Therefore,

$$m^*(\mathcal{O} \cap E) + m^*(\mathcal{O} \cap E^c) \leq \sum_{k=1}^{\infty} m^*((a_k, b_k) \cap E) + \sum_{k=1}^{\infty} m^*((a_k, b_k) \cap E^c) < m^*(E) + \epsilon.$$

Since E is finite,

$$\begin{aligned} m^*(\mathcal{O} \cap E) + m^*(\mathcal{O} \cap E^c) &< m^*(E) + \epsilon \\ m^*(E) + m^*(\mathcal{O} \setminus E) &< m^*(E) + \epsilon \\ m^*(\mathcal{O} \setminus E) &< \epsilon \end{aligned}$$

and thus E is measurable by Theorem 11. □

Problem (21)**.** Use property (*ii*) of Theorem 11 as the primitive defninition of a measurable set and prove that the union of two measurable sets is measurable. Then do the same with property (*iv*).

Proof. Let E_1, E_2 be measurable sets. Let G_1, G_2 be G_δ sets such that $G_1 \supset E_1$ and $G_2 \supset E_2$, where $m^*(G_1 \setminus E_1) = 0$ and $m^*(G_2 \setminus E_2) = 0$.

Note that $(G_1 \cup G_2) \setminus (E_1 \cup E_2) = (G_1 \setminus (E_1 \cup E_2)) \cup (G_2 \setminus (E_1 \cup E_2))$. Thus,

$$\begin{aligned} m^*((G_1 \cup G_2) \setminus (E_1 \cup E_2)) &= m^*((G_1 \setminus (E_1 \cup E_2)) \cup (G_2 \setminus (E_1 \cup E_2))) \\ &\leq m^*((G_1 \setminus (E_1 \cup E_2))) + m^*((G_2 \setminus (E_1 \cup E_2))) \\ &\leq m^*(G_1 \setminus E_1) + m^*(G_2 \setminus E_2) \\ &\leq 0 + 0 = 0 \end{aligned}$$

and thus $m^*(E_1 \cup E_2) = 0$.

Now, for E_1 and E_2 measurable, by property (*iv*) there are F_σ sets F_1 and F_2, where $F_1 \subset E_1$ and $F_2 \subset E_2$, such that $m^*(E_1 \setminus F_1) = 0$ and $m^*(E_2 \setminus F_2) = 0$. Note that $(E_1 \cup E_2) \setminus (F_1 \cup F_2) \subseteq (E_1 \setminus F_1) \cup (E_1 \setminus F_2)$. Therefore,

$$\begin{aligned} m^*((E_1 \cup E_2) \setminus (F_1 \cup F_2)) &\leq m^*(E_1 \setminus F_1) + m^*(E_1 \setminus F_2) \\ &\leq 0 + 0 = 0 \end{aligned}$$

Thus, $E_1 \cup E_2$ is measurable. □

Problem (22)**.** For any set A, define $m^{**}(A) \in [0, \infty]$ by $m^{**}(A) = \inf\{m^*(\mathcal{O})|\mathcal{A} \subset \mathcal{O}, \mathcal{O}open\}$. How is m^{**} related to m^*. They are equal.

Proof. First, we will show $m^*(A) \leq m^{**}(A)$. Let $\mathcal{O}$ be an open set such that $A \subset \mathcal{O}$. Any open set can be written as the xxx of disjoint open intervals I_j. Since $\mathcal{O}$ is measurable by σ-additivity, $m(\mathcal{O}) = \sum_{j=1}^{\infty} m(I_j) = \sum_{j=1}^{\infty} \ell(I_j)$. (Note that $m^*(\mathcal{O}) = m(\mathcal{O})$). Since $A \subset \mathcal{O} = \bigcup_{j=1}^{\infty} I_j$, then $\{I_j\}$ is a covering with open intervals of A, then

$$m^*(A) = \text{glb}\left\{\sum_{k=1}^{\infty} \ell(I_k)|A \subset \bigcup_{k=1}^{\infty} I_k\right\} \leq m(\mathcal{O}),$$

so $m^*(A)$ is a lower bound of $\{m^*(\mathcal{O})|A \subset \mathcal{O}\}$, then $m^*(A) \leq m^{**}(A)$.

Now we will show $m^{**}(A) \leq m^*(A)$. We consider a covering of A with open intervals such that $\sum_{j=1}^{\infty} \ell(I_j) \leq m^*(A) + \epsilon$. Now, $\bigcup_{j=1}^{\infty} I_j = \mathcal{O}$, an open set, and

$$m^{**}(A) \leq m(\mathcal{O}) \leq \sum_{j=1}^{\infty} m(I_j),$$

by subadditivity, and

$$\sum_{j=1}^{\infty} m(I_j) = \sum_{j=1}^{\infty} \ell(I_j) \leq m^*(A) + \epsilon$$

for an arbitrary ϵ. This implies that $m^{**}(A) \leq m^*(A)$. □

Problem (23)**.** For any set A, define $m^{***}(A) \in [0, \infty]$ by $m^{***}(A) = lub\{m^*(\mathcal{F})|\mathcal{F} \subset A, \mathcal{F}closed\}$. How is m^{***} related to m^*.

Proof. We will prove that $m^{***} \leq m^*$. Since $\mathcal{F}$ is closed, $\mathcal{F}$ is measurable. Since $\mathcal{F} \subset A$, by monotonicity $m^*(\mathcal{F}) \leq m^*(A)$. Then, $m^*(A)$ is an upper bound of $m^*(F)$ for all $F \subset A$, then $lub\{m^*(F)\} \leq m^*(A)$. Then, $m^{***} \leq m^*$.

And example where inner measure is strictly less than outer measure is the Vitali set (or, any nonmeasurable set). □

Problem (24)**.** Show that if E_1 and E_2 are measureable then $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$.

Proof. If $m(E_1) = \infty$ or $m(E_2) = \infty$, then the equality holds. So, assume $m(E_1)$ and $m(E_2)$ are finite. We know from set theory that $E_1 \cup E_2 = E_1 \setminus (E_1 \cap E_2) \cup (E_1 \cap E_2) \cup E_2 \setminus (E_1 \cap E_2)$. Then,

$$m(E_1 \cup E_2) = m(E_1 \setminus (E_1 \cap E_2)) + m(E_1 \cap E_2) + m(E_2 \setminus (E_1 \cap E_2))$$

and by the excision property,

$$m(E_1 \cup E_2) = m(E_1) - m(E_1 \cap E_2) + m(E_1 \cap E_2) + m(E_2) - m(E_1 \cap E_2)$$

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

□

Problem (25)**.** Show that $m(B_1) < \infty$ is necessary in Theorem 15, part (*ii*), of the theorem regarding continuity of measure.

Proof. Proceed by contradiction. Note that $B_n = [n, \infty)$, $\bigcap_{n=1}^{\infty} B_n = \emptyset$, $m(\emptyset) = 0$. So consider the measure of each B_n individually, $m(B_n) = \infty$. Then, $\lim_{n\to\infty} m(B_n) = \infty$. Contradiction. □

Problem (26)**.** Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable sets. Prove that for any set A,

$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*(A \cap E_k)$$

Proof. We know that for any j, $E_j \subseteq \bigcup_{j=1}^{\infty} E_j$. Thus, $A \cap E_j \subseteq A \cap \bigcup_{j=1}^{\infty} E_j$. By monotonicity,

$$\sum_{j=1}^{\infty} m^*(A \cap E_j) \leq m^*\left(A \cap \bigcup_{j=1}^{\infty} E_j\right).$$

To see the inequality in the other direction, observe that by σ-subadditivity,

$$m^*\left(A \cap \bigcup_{j=1}^{\infty} E_j\right) = m^*\left(\bigcup_{j=1}^{\infty}(A \cap E_j)\right) \leq \sum_{j=1}^{\infty} m^*(A \cap E_j)$$

by subadditivity. □

Problem (27)**.** Let $\mathcal{M}'$ be any σ-algebra of subsets of $\mathbb{R}$ and m' a set function on $\mathcal{M}'$ which takes values in $[0, \infty]$, is countably additive, and such that $m'(\emptyset) = 0$.

- (i) Show that m' is finitely additive, monotone, countably monotone, and posseses the excision property.
- (ii) Show that m' possesses the same continuity properties as Lebesgue measure.

Proof.
- (i) Finitely additive: Consider disjoint, finite sets $E_1, E_2, \ldots, E_N, \emptyset, \emptyset, \ldots$. Then, by σ-additivity,

$$m'\left(\bigcup_{j=1}^{N} E_j\right) = \sum_{j=1}^{N} m'(E_j)$$

Monotonicity: Assume $A \subset B$. Then, $B = A \cup B \setminus A$. By σ-additivity, $m(B) = m(A) + m(B \setminus A)$. But, $m(B \setminus A) \geq 0$, which implies that $m(A) \leq m(B)$.

Countable monotonicity: Countable monotonicity states that

$$m(E) = m\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m(E_j),$$

where $\{E_j\}$ is a countable collection of sets in $\mathcal{M}'$. We disjoint the sets. Say

$$\begin{aligned} B_1 &= E_1 \\ B_2 &= E_2 - B_1 \\ &\vdots \\ B_N &= E_N - \left(\bigcup_{j=1}^{N-1} B_j\right) \\ &\vdots \end{aligned}$$

where $\bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty} E_j$. Now, the $\{B_j\}$ are disjoint by σ-additivity of m':

$$m'\left(\bigcup_{j=1}^{\infty} B_j\right) = \sum_{j=1}^{\infty} m'(B_j) \leq \sum_{j=1}^{\infty} m(E_j).$$

Excision property: The excision property states that if $A \subset B$, then $B = A \cup (B \setminus A)$. Assume $m'(A) < \infty$. If $m'(B) = \infty$, then $m'(B \setminus A) = \infty$. Then since $m'(B) = m'(A) + m'(B \setminus A)$, $m'(B) - m'(A) = m(B \setminus A)$. If $m'(B) < \infty$, then by subadditivity $m'(B) = m'(A) + m'(B \setminus A)$. Then $m'(B) - m'(A) = m'(B \setminus A)$.

□

Problem (28)**.** Show that continuity of measure together with finite additivity of measure implies countable additivity of measure (σ-additivity).

Proof. Let $\{E_j\}$ be a disjoint collection of measurable sets. Let $C_n = \bigcup_{j=1}^{N} E_j$, and note that $C_N \subset C_{N+i}$ (increasing) and $\bigcup_{N=1}^{\infty} C_N = \bigcup_{j=1}^{\infty} E_j$. Then, by continuity from below,

$$= m\left(\bigcup_{j=1}^{\infty} E_j\right) = m\left(\bigcup_{N=1}^{\infty} C_N\right) = \lim m(C_N)$$

from which we see that

$$\lim m(C_N) = \lim_{N\to\infty} m\left(\bigcup_{j=1}^{N} E_j\right) = \lim_{N\to\infty} \sum_{j=1}^{N} m(E_j) = \sum_{j=1}^{N} m(E_j).$$

□

Problem (29)**.**

- (i) Show that rational equivalence defines an equivalence relation on any set.
- (ii^*) Explicitly find a choice set for the rational equivalence relation on $\mathbb{Q}$.
- (iii) Define two numbers to be irrationally equivalent provided their difference is irrational. Is this an equivalence relation on $\mathbb{R}$? Is this an equivalence relation on $\mathbb{Q}$?

Proof.

- (i) Consider a set A, where $x, y \in A$, and say $x \sim y$ if $x - y$ is rational. We must check that $\sim$ is reflexive, symmetric, and transitive.
  - Reflexive: $x - x = 0 \in \mathbb{Q}$.
  - Symmetric: If $x - y = r \in \mathbb{Q}$, then $y - x = -r \in \mathbb{Q}$.
  - Transitive: Assume $x - y = r_1 \in \mathbb{Q}$ and $y - z = r_2 \in \mathbb{Q}$. Then, $x - z = x - y + y - z = r_1 + r_2 \in \mathbb{Q}$.
- (ii^*) As seen in the proof of Theorem 17, the choice set for the rational equivalence relation on $\mathbb{Q}$ is not measurable. Thus, we cannot explicitly state it. However, we know that two numbers will be in the same equivalence class if they are non-repeating decimals that eventually match up.
- (iii) No. In both $\mathbb{R}$ or $\mathbb{Q}$, this fails to be reflexive.

□

Problem (30)**.** Show that any choice set for the rational equivalence relation on a set of positive outer measure must be uncountably infinite.

Proof. Assume to the contrary that there is a choice set for the rational equivalence relation on a set of positive outer measure that is not uncountable. Then, it is either countably infinite or finite. In either case, it is measurable–a contradiction. □

Problem (31)**.** Justify the assertion in the proof of Vitali's Theorem that it suffices to consider the case that E is bounded.

Proof. Note that $E = \bigcup_{n=1}^{\infty} E \cap (-n, n)$. By the subadditivity of outer measure,

$$0 < m^*(E) = m^*\left(\bigcup_{n=1}^{\infty} E \cap (-n, n)\right) \leq \sum_{n=1}^{\infty} m^*(E \cap (-n, n)).$$

Thus, there is an n_0 such that $m^*(E \cap (-n_0, n_0)) > 0$. Thus, showing that the bounded set $E \cap (-n_0, n_0)$ contains a subset that fails to be measurable is sufficient to show that E contains a subset that fails to be measurable. □

Problem (32)**.** Does Lemma 16 remain true if Λ is allowed to be finite or to be uncountably infinite? Does it remain true if Λ is allowed to be unbounded?

Lemma 16: Let E be a bounded measurable set of real numbers. Suppose there is a bounded countably infinite set of real numbers Λ for which the collection of translates of E, $\{\lambda + E\}_{\lambda \in \Lambda}$ is disjoint. Then, $m(E) = 0$.

Proof. We know that $\bigcup_{\lambda \in \Lambda} \lambda + E$ is bounded. Then, $m\left(\bigcup_{\lambda \in \Lambda} \lambda + E\right) < \infty$. However,

$$m\left(\bigcup_{\lambda \in \Lambda} \lambda_j + E\right) = \sum_{j=1}^{\infty} m(\lambda_j + E).$$

Since m is translation invariant,

$$m\left(\bigcup_{\lambda \in \Lambda} \lambda_j + E\right) = \sum_{j=1}^{\infty} m(\lambda_j + E) = \sum_{j=1}^{\infty} m(E),$$

which implies that $m(E) = 0$. □

Problem (33)**.** Let E be a nonmeasurable set of finite outer measure. Show that there is a G_δ set G that contains E for which $m^*(E) = m^*(G)$ while $m^*(G \setminus E) > 0$.

Proof. (Alexander's proof)
Note that if $m^*(E) = m^*(G)$ then $m^*(G) - m^*(E) = 0$. So we can restate the problem to want to show that $m^*(G \setminus E) > m^*(G) - m^*(E)$. Assume to the contrary that $m^*(G \setminus E) \leq m^*(G) - m^*(E)$.

Let $\{I_{k,n} = \{I_{k,n}\}_{k=1}^{\infty}\}_{n=1}^{\infty}$ be a collection of open coverings of E, and let $I_n = \bigcup_{k=1}^{\infty} I_{k,n}$. Note that $m^*(I_n) \leq \sum_{k=1}^{\infty} \ell(I_n) \leq m^*(E) + \epsilon$ for $\epsilon > 0$. Now, let $G = \bigcap_{n=1}^{\infty} I_n$. Then, G is a G_δ set and $E \subseteq G$. Observe for any n:

$$m(G) \leq m^*\left(\bigcap_{n=1}^{\infty} I_n\right) \leq m^*(I_n) < m^*(E) + \epsilon$$

and thus $m^*(G) \leq m^*(E) + \epsilon$. Finally, observe that

$$m^*(G \setminus E) \leq m^*(G) - m^*(E) < \epsilon,$$

which implies that E is measurable, a contradiction. □

Problem (34)**.** Show there is a continuous, strictly increasing function on the interval $[0,1]$ that maps a set of positive measure onto a set of measure zero.

Proof. We consider $\psi_0 : [0,1] \to [0,1]$ defined as $\psi_0(x) = \frac{1}{2}\psi = \frac{1}{2}[\phi(x) + x]$, a function that is continuous and strictly increasing. This function is invertable.

Let $\psi^{[-1]}(B)$ denote inverse image, and $\psi^{(-1)}(B)$ denote the inverse function. Since ψ is one to one, etc, the inverse image and the inverse function are the same. Note that $[\psi_0^{(-1)}]^{[-1]}(\mathcal{O}) = [\psi_0^{(-1)}]^{(-1)}(\mathcal{O}) = \psi_0(\mathcal{O})$.

Note that $m(\psi_0(c)) = \frac{1}{2}$. Also note that $\psi_0^{(-1)}(\psi_0(c)) = c$. Thus, $m(\psi_0^{(-1)}(\psi_0(c))) = m(c) = 0$. □

Problem (38)**.** Let the function $f : [a,b] \to \mathbb{R}$ be Lipschitz, that is, there is a constant $c \geq 0$ such that for all $u, v \in [a,b]$, $|f(u) - f(v)| \leq c|u - v|$. Show that f maps a set of measure zero onto a set of measure zero. Show that f maps an F_σ set onto an F_σ set. Conclude that f maps a measurable set to a measurable set.

Proof. □

2 Chapter 3

Problem (1)**.** Suppose f and g are continuous functions on $[a,b]$. Show that if $f = g$ almost everywhere on $[a,b]$, then, in fact, $f = g$ on $[a,b]$. Prove that a similar assertion is not true if $[a,b]$ is replaced by a general measurable set E.

Proof. Let $N = \{x \in [a,b] : f(x) \neq g(x)\}$. Since $f = g$ almost everywhere on $[a,b]$, $m(N) = 0$.

Let $x \in N$ and $n \in \mathbb{N}$, and consider a sequence of intervals $I_n = \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$, together with $\left(a, a + \frac{1}{n}\right)$ and $\left(b - \frac{1}{n}, b\right)$. Note that each I_n has positive outer measure, and when n is large enough, $I_n \cap A \neq \emptyset$ and $I_n \cap N = \{x\}$. Consider a sequence $x_n \in I_n$. Then, when n is large enough, $x_n \notin N$, $x_n \in A$, and $\lim x_n = x$. Since $f(x_n) = g(x_n)$,

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x).$$

Now, consider two functions f and g with domains in $\mathbb{Z}$ defined by $f(x) = 1$ and $g(x) = 0$ for all $x \in \mathbb{Z}$. Then, since $m(\mathbb{Z}) = 0$, $f = g$ almost everywhere on $\mathbb{Z}$. However, f and g certainly are not equal. Thus, the assertion does not hold for all sets. □

Problem (2)**.** Let D and E be measurable sets and f a function with domain $D \cup E$. We proved that f is measurable on $D \cup E$ if and only if its restrictions to D and E are measurable. Show that the same is not true if "measurable" is replaced by "continuous."

Proof. Consider a function f, defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in [0,1] \\ 1 & \text{if } x \in (1,2] \end{cases}$$

Note that f is continuous on $[0,1]$ and f is continuous on $(1,2]$. However, f is discontinuous at $x = 1$ on $[0,2]$. □

Problem (3)**.** Suppose a function f has a measurable domain and is continuous except at a finite number of points. Prove that$f$ is measurable.

Proof. Let $B = \{x : f(x) \text{ is not continuous}\}$. Since B is finite, f is measurable on B and $m(B) = 0$. Consider $A = E \setminus B$. Since f is continuous on A and A is measurable, by Proposition 3, f is measurable on A. Thus f is measurable on $A \cup B = (E \setminus B) \cup B = E$. □

Problem (4)**.** Suppose f is a real-valued function on $\mathbb{R}$ such that $f^{-1}(c)$ is measurable for each number c. Show that f is not necessarily measurable.

Proof. Let W be the Vitali set. Let f be a function on W defined by $f(x) = x$. Then, $f^{-1}(x) = x$ is measurable. However, the domain W is not measurable, and thus f is not measurable. □

Problem (5)**.** Suppose the function f is defined on a measurable set E and has the property that $\{x \in E : f(x) > c\}$ is measurable for each rational number c. Prove that f is measurable.

Proof. (Method I-Gatto)
Let $c \in \mathbb{Q}$ and $r \in \mathbb{R}$. Note that

$$\{x : f(x) > r\} = \bigcup_{c>r} \{x : f(x) > c\}$$

Since the right hand side is a union of measurable sets, the union is measurable. Thus, the left hand side is measurable. Therefore, f is measurable. □

Proof. (Method II-Christina and Steve)
Let $r \in \mathbb{R}$, and let c_n be a sequence of rational numbers such that $c_n \to r$. Then,

$$\{x \in E : f(x) > r\} = \bigcup_{n=1}^{\infty} \{x \in E : f(x) > c_n\}$$

and measurability of the right side implies measurability of the left side. Thus, f is measurable. □

Problem (6)**.** Let f be a function with measurable domain D. Show that f is measurable if and only if the fuction g defined on $\mathbb{R}$ by $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.

Proof. (Method I-Christina)
" $\Rightarrow$ " Let f be measurable on D, and let g be defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in D \\ 0 & \text{if } x \notin D \end{cases}$$

Then, g is measurable on D and. Consider g on $\mathbb{R} \setminus D$.

- If $a \geq 0$, then $\{x \in \mathbb{R} \setminus D : g(x) > a\} = \emptyset$, which is measurable
- If $a < 0$, then $\{x \in \mathbb{R} \setminus D : g(x) > a\} = \mathbb{R} \setminus D$, which is measurable

Thus, g is measurable on $\mathbb{R} \setminus D$. Therefore, g is measurable on $\mathbb{R} = (\mathbb{R} \setminus D) \cup D$.

" $\Leftarrow$ " Assume g is measurable on $\mathbb{R}$. Then, g is measurable on $D \subset \mathbb{R}$. Since $g = f$ on D, this is equivalent to stating that f is measurable on D. □

Proof. (Method II-Gatto)
Since f is measurable, note that for $\alpha \geq 0$,

$$\{x \in D : f(x) > \alpha\} = \{x : g(x) > \alpha\},$$

and for $\alpha < 0$,

$$\{x : g(x) > \alpha\} = \{x \in D : f(x) > \alpha\} \cup \{\mathbb{R} \setminus D\}$$

all of which are measurable sets, and thus g is measurable on its domain.

Conversely, if g is measurable, the same equalities above hold, and thus f is measurable. □

Problem (7)**.** Let the function f be defined on a measurable set E. Show that f is measurable if and only if for each Borel set A, $f^{-1}(A)$ is measurable.

Proof. " $\Rightarrow$ " Consider $\mathcal{A}$, the σ-algebra with the property that $f^{-1}(U)$ is measurable for each $U \in \mathcal{A}$. By Proposition 2, since f is measurable, $f^{-1}(\mathcal{O})$ is measurable for every open set $\mathcal{O}$. Thus, $\mathcal{O} \in \mathcal{A}$ for every open set $\mathcal{O}$. Therefore, Borel sets lie in $\mathcal{A}$.

" $\Leftarrow$ " Assume that for every Borel set A, $f^{-1}(A)$ is measurable. Since Borel sets contain every open set, by Proposition 2, f is measurable. □

Problem (8)**.**

Proof. □

Problem (9)**.** Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E. Define E_0 to be the set of points $x \in E$ at which $\{f_n(x)\}$ converges. Prove that E_0 is measurable.

Proof. □

Problem (10)**.** Suppose f and g are real-valued functions defined on all of $\mathbb{R}$, f is measurable, and g is continuous. Show that the composition $f \circ g$ is not necessarily measurable.

Proof. Let $f = \chi_E$ and $W = \psi(E)$, where $g = \psi^{-1}$, the Cantor-Lebesgue function defined in Proposition 21 of chapter 2. The function ψ^{-1} is continuous and therefore is measurable. Compute:

$$(f \circ g)^{-1}(1) = g^{-1}(f^{-1}(1)) = g^{-1}(E) = \psi(E) = W$$

but, by Propisition 21 of Chapter 2, W is a non-measurable set. □

Problem (11)**.** Let f be measurable and g one-to-one from $\mathbb{R}$ onto $\mathbb{R}$ which has a Lipschitz inverse. Show that the composition $f \circ g$ is measurable.

Proof. □

Problem (12)**.** Let f be a bounded measurable function on E. Show that there are sequences of simple functions on E, $\{\phi_n\}$ and $\{\psi_n\}$, sich that $\{\phi_n\}$ is decreasing and each of these sequences converges to f uniformly on E.

Proof. (Christina's proof)
Let $n > 0$. Then, by the Simple Approximation Lemma, for each $n > 0$ there are simple functions $\phi_{\frac{1}{n}}$ and $\psi_{\frac{1}{n}}$ such that $\phi_{\frac{1}{n}} \leq f \leq \psi_{\frac{1}{n}}$ and $\psi_{\frac{1}{n}} - \phi_{\frac{1}{n}} < \frac{1}{n}$ on E. Next, let

$$\phi'_{\frac{1}{n}} = \max_{k>n} \left\{ \phi_{\frac{1}{n}} \right\}$$

and

$$\psi'_{\frac{1}{n}} = \max_{k>n} \left\{ \psi_{\frac{1}{n}} \right\}.$$

Observe that $\phi'_{\frac{1}{n}}$ increases, while $\psi'_{\frac{1}{n}}$ decreases. Furthermore, observe that

$$\frac{1}{n} > f - \phi_{\frac{1}{n}} \geq f - \phi'_{\frac{1}{n}}$$

and

$$\frac{1}{n} > \psi_{\frac{1}{n}} - f \geq \psi'_{\frac{1}{n}} - f$$

and thus

$$\lim_{n\to\infty} f - \phi'_{\frac{1}{n}} < \frac{1}{n}$$

and

$$\lim_{n\to\infty} \psi'_{\frac{1}{n}} - f < \frac{1}{n}$$

independently of x, and thus the sequences $\left\{\psi'_{\frac{1}{n}}\right\}$ and $\left\{\phi'_{\frac{1}{n}}\right\}$ converge to f uniformly. □

Problem (13)**.** A real-valued measurable function is said to be semisimple provided it takes only a countable number of values. Let f be any measurable function on E. Show that there is a sequence of semisimple functions $\{f_n\}$ on E that converges to f uniformly on E.

Proof. (Christina's proof)
Let f be a measurable function on E. Consider a set E_j for every integer j, where $E_j = \left\{x \in E : \frac{j-1}{n} \leq f(x) < \frac{j}{n}\right\}$. Then, define a function

$$f_n = \sum_{j\in\mathbb{Z}} \frac{j}{n} \chi_{E_j}.$$

Since $\mathbb{Z}$ is countable, f_n is a semisimple function. Consider the sequence of functions $\{f_n\}$ and let $\epsilon > 0$. Then, there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Take $n > N$. Then,

$$|f - f_n| < \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

and thus $\{f_n\}$ converges uniformly to f. □

Proof. (Alexander's proof)
Divide the real line into subintervals of length 2^{-n} as follows: let $I_k = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$ for $k \in \mathbb{Z}$. Now, let $E_k = f^{-1}(I_k)$, and let

$$f_n = \sum_{k=1}^{\infty} \frac{k-1}{2^n} \chi_{E_k}.$$

Note that f_n is semisimple. Furthermore, note that as $n \to \infty$, $f_n \to f$ uniformly, since for all $n > N$, $|f - f_n| < \frac{1}{2^n} < \epsilon$. □

Problem (14)**.** Let f be a measurable function that is finite a.e. on E and $m(E) < \infty$. For each $\epsilon > 0$, show that there is a measurable set F contained in E such that f is bounded on F and $m(E \setminus F) < \epsilon$.

Proof. Remember that $f = f^+ + f^-$. First, consider f^+. Let $N = \{x \in E : f(x) = +\infty\}$, and note that $m(N) = 0$. Consider $E_n = \{x \in E : |f(x)| > n\}$. Then,

$$\bigcap_{n=1}^{\infty} E_n = N.$$

Note that $m(N) = \lim_{n=1}^{\infty} m(E_n) = \lim_{n=1}^{\infty} m\left(\{x \in E : |f(x)| > n\}\right) = 0$. Then, given any $\epsilon > 0$, there is a set such that $m\left(E_n\right) < \epsilon$.

Define a set $F_n = \{x \in E : |f(x)| \leq n\} = E_n^c$. Then, F_n is bounded, and

$$m(E \setminus F_n) = m(F_n^c) = m(E_n) < \epsilon.$$

Proceed similarly to see that the inequality holds for f^-. □

Problem (15)**.** Let f be a measurable function on E that is finite a.e. on E and $m(E) < \infty$. Show that for each $\epsilon > 0$, there is a measurable set F contained in E and a sequence $\{\phi_n\}$ of simple functions on E such that $\{\phi_n\} \to f$ uniformly on F and $m(E \setminus F) < \epsilon$.

Proof. Define F_n the same as in the previous problem, $F_n = \{x \in E : |f(x)| \leq n\}$. Then, as seen in the previous exercise, $m(E \setminus F_n) < \epsilon$ for any given ϵ whenever $n > N_\epsilon$. Furthermore, by exercise 12 in this section, since f is bounded on F there is a sequence $\{\phi_n\}$ of simple functions on E such that $\{\phi_n\} \to f$. □

Problem (16)**.** Let I be a closed, bounded interval and E a measurable subset of I. Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which

$$h = \chi_E \text{ on } F \text{ and } m(I \setminus F) < \epsilon.$$

Proof. Without loss of generality, Let $E \subseteq [0,1] = I$. Let $\mathcal{O} = \{\mathcal{O}_j\}_{j=1}^{\infty}$ be an open disjoint covering of E such that $\sum_{j=1}^{\infty} \ell(\mathcal{O}_j) \leq m(E) + \frac{\epsilon}{2}$. Let $F = [0,1] \setminus (\mathcal{O} \setminus E)$. Since F is a collection of intervals, F is a measurable subset of $[0,1]$. Furthermore, $I = F \cup (\mathcal{O} \setminus E)$, thus

$$\begin{aligned} m(I) &= m(F \cup (\mathcal{O} \setminus E)) \\ m(I) &= m(F) + m(\mathcal{O} \setminus E) \\ m(I) - m(F) &= m(\mathcal{O} \setminus E) < \epsilon \end{aligned}$$

and thus $m(I \setminus F) < \epsilon$. □

Problem (17)**.** Let I be a closed, bounded interval and ψ a simple function defined on I. Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which

$$h = \psi \text{ and } m(I \setminus F) < \epsilon.$$

Proof. (Warning: this proof is from Gatto's notes and is not clear)
By number 16, there is a step function $h_i(x)$ such that $h_i(x) = \chi_{E_i}(x)$ except on a set $B_i = I \setminus F_n$, where $m(B_i) < \frac{\epsilon}{N}$.

Note that $F_n = [0,1] \setminus ((\mathcal{O} \setminus E) \cup (E \setminus \mathcal{O}))$

Let $h = \sum_{i=1}^{N} a_i h_i(x) = \sum_{i=1}^{N} a_i \chi_{E_i}(x)$ except at $B = \bigcup_{i=1}^{N} B_i$ but $m(B) \leq \sum_{i=1}^{N} m(B_i) < \epsilon$.

Note that $F = I \setminus B$, then $I \setminus F = I \cap \left((I \cap B)^C\right)^C = I \cap (I^C \cap B) = I \cap B = B$.

Finally, $m(I \setminus F) = m(B) \leq \epsilon$. □

Problem (18)**.** Let I be a closed, bounded interval and f a bounded measurable function defined on I. Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which

$$|h - f| < \epsilon \text{ on } F \text{ and } m(I \setminus F) < \epsilon.$$

Proof. □

Problem (19)**.** Show that the sum and product of two simple functions are simple, as are the max and the min.

Proof. (Christina's proof)
Let $\phi(x)$ and $\psi(x)$ be simple functions on E. Then,

$$\phi(x) = \sum_{k=1}^{n} a_k \chi_{E_k}, \text{ where } E_k = \{x \in E : \phi(x) = a_k\}$$

$$\psi(x) = \sum_{j=1}^{m} b_j \chi_{A_j}, \text{ where } A_j = \{x \in E : \psi(x) = b_j\}$$

Next, define new sets $B_{p,q} = E_p \cap A_q$ and $c_{p,q} = a_p + b_q$.Then,

$$\phi(x) + \psi(x) = \sum_{j,k=1}^{p,q} a_j b_k \chi_{B_{jk}}$$

The max and min of two different simple functions must be simple, since they still will be taking only a finite number of values. □

Problem (20)**.** Let A and B be any two sets. Show that

$$\begin{aligned} \chi_{A\cap B} &= \chi_A \cdot \chi_B \\ \chi_{A\cup B} &= \chi_A + \chi_B - \chi_A \cdot \chi_B \\ \chi_{A^C} &= 1 - \chi_A. \end{aligned}$$

Proof. (Alexander)
Note that

$$\chi_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \qquad \text{and } \chi_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

and thus

$$\chi_A \chi_B = \begin{cases} 1 & \text{if } x \in A \text{ and } x \in B \\ 0 & \text{if } x \notin A \text{ and } x \in B \\ 0 & \text{if } x \notin A \text{ and } x \notin B \\ 0 & \text{if } x \notin A \text{ and } x \notin B \end{cases}$$

Or, more simply, $\chi_A \chi_B = \chi_{A\cap B}$. The other two properties follow similarly. □

Problem (21)**.** For a sequence $\{f_n\}$ of measurable functions with common domain E, show that each of the following functions is measurable:

$$\inf\{f_n\}, \sup\{f_n\}, \liminf\{f_n\}, \limsup\{f_n\}$$

Proof. (Alexander's proof)
Note that for every $c \in \mathbb{R}$,

$$\inf\{f_n\} = \{x \in E : \inf\{f_n\} \leq c\} = \bigcup_{n=1}^{\infty} \{x \in E : f_n(x) \leq c\}$$
$$\sup\{f_n\} = \{x \in C : \sup\{f_n\} \geq c\} = \bigcap_{n=1}^{\infty} \{x \in E : f_n(x) \geq c\}$$

The elements on the far right side of the equations are all measurable, and unions of measurable sets are measurable, thus the inf and sup are measurable. Furthermore, observe that

$$\lim_{n\to\infty} \inf\{f_n\} = \sup_n \inf_{k\geq n} \{f_k\}$$
$$\lim_{n\to\infty} \sup\{f_n\} = \inf_n \sup_{k\geq n} \{f_k\}$$

and thus the lim sup and lim inf are measurable. □

Problem (22)**.** Let $\{f_n\}$ be an increasing sequence of continuous functions on $[a, b]$ which converges pointwise on $[a, b]$ to the continuous function f on $[a, b]$. Show that the convergence is uniform on $[a, b]$.

Proof. (Proof by Christina, Jamie, Alexander)
Let $\epsilon > 0$, and define $E_n = \{x \in [a, b] : f(x) - f_n(x) < \epsilon\}$. We want E_n to be open, so note that $f - f_n(x)$ is continuous and thus E_n is open relative to $[a, b]$.

Since $f_n \to f$ pointwise, then there is an $N \in \mathbb{N}$ such that for all $n > N$, $|f(x) - f_n(x)| < \epsilon$. Let $n_0 > N$, and consider $x \in E_{n_0}$. Then, $x \in \bigcup_{n=1}^{\infty} E_n$. Thus, $[a, b] \subseteq \bigcup_{n=1}^{\infty} E_n$. In other words, $\{E_n\}$ is an open covering of $[a, b]$.

Since $\{E_n\}$ is an open covering of $[a, b]$ and since $[a, b]$ is compact, by Heine-Borel there is a finite subcovering of $[a, b]$. Denote this subcovering $\{E_{n_1}, E_{n_2}, \ldots, E_{n_k}\}$. Let $N_0 = \max\{n_1, n_2, \ldots, n_k\}$. Let $n \geq N_0$ and $x \in [a, b] \subseteq E_{n_1} \cup E_{n_2} \cup \cdots \cup E_{n_k}$. Thus, $x \in E_{n_i}$ for some i. However, $n > N_0 > n_i$, and f_n converges pointwise, thus

$$|f(x) - f_n(x)| < |f(x) - f_{n_i}(x)| < \epsilon$$

and thus we have uniform convergence. □

Problem (23)**.** Express a measurable function as the difference of nonnegative measurable functions and thereby prove the general Simple Approximation Theorem based on the special case of a nonnegative measurable function.

Proof. Remember that $f = f^+ - f^-$, where $f^+ = \{\max\{f(x), 0\}\}$ and $f^- = \{\max\{-f(x), 0\}\}$. Both f^+ and f^- are measurable and thus we have satisfied all of the needed contditions. □

Problem (24)**.** Let I be an interval and $f : I \to \mathbb{R}$ be increasing. Show that f is measurable by first showing that, for each number n, the strictly increasing function $x \mapsto f(x) + \frac{x}{n}$ is measurable, and then taking pointwise limits.

Proof. (Christina)
Note that there is an x_0 such that $f(x_0) + \frac{x_0}{n} > \alpha$ since f is increasing. Then, $(x_0, b] \subseteq \{x \in I : f(x) + \frac{x}{n} > \alpha\}$, and thus $f(x) + \frac{x}{n}$ is measurable on its domain. Therefore, we will consider the sequence functions $f_n = f(x) + \frac{x}{n}$. Then, $f_n \to f$, and all f_n are measurable, thus f is measurable by proposition 9. □

3 Chapter 4

Problem (9)**.** Let E have measure zero. Show that if f is a bounded function on E, then f is measurable and $\int_E f = 0$.

Proof. Let f be a function, and consider the set $\{x \in E : f(x) > \alpha\}$, where $\alpha \in \mathbb{R}$. But, $\{x \in E : f(x) > \alpha\} \subset E$. So, by monotonicity, $m\{x \in E : f(x) > \alpha\} \leq m(E) = 0$.

Now, consider a simple function ψ on E such that:

$$\int_E f = \inf_{f \leq \psi} \int_E \psi$$

Any simple function $\psi = \sum_{i=1}^n a_i \chi_{E_i}$ on E means $E_i \subset E$ and thus $\int_E \psi = 0$. Thus,

$$\int_E f = \inf_{f \leq \psi} \int_E \psi = 0.$$

□

Problem (10)**.** Let f be a bounded measurable function on a set of finite measure E. For a measurable subset A of E, show that $\int_A f = \int_E f \cdot \chi_A$.

Proof. Observe that $\int_E f\chi_A = \int_{E\setminus A} f\chi_A + \int_A f\chi_A$. Note that $\int_{E\setminus A} f\chi_A = 0$, since χ is zero in A^c. (This is because for any Y, $-\frac{1}{n}\chi_Y \leq 0 \leq \frac{1}{n}\chi_Y$, thus $\int -\frac{1}{n}\chi_Y \leq \int 0 \leq \int \frac{1}{n}\chi_Y$). All together, we see that

$$\int_E f\chi_A = \int_{E\setminus A} f\chi_A + \int_A f\chi_A = \int_A f\chi_A = \int_A f$$

□

Problem (11)**.** Show that the Bounded Convergence Theorem does not hold for the Riemann integral.

Proof. Consider an enumeration of $\mathbb{Q} = \{r_n\}_{n=1}^\infty$ in $[0, 1]$, and define

$$f_R(x) = \begin{cases} 1 & \text{if } x = r_i, 1 \leq i \leq k \\ 0 & \text{when } x \text{ is odd} \end{cases}$$

And note that

$$\lim_{k\to\infty} f_R(x) = f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0 & whenx \notin \mathbb{Q} \cap [0, 1] \end{cases}$$

Now consider the Riemann upper sum. Note that any Riemann upper sum is 1, ie, $U(f, P) = 1$, and any lower sum is zero, ie, $L(f, P) = 0$. Thus, $R\int f$ does not exist, since the infimum of upper sums and supremum of lower sums are not equal. □

Problem (12)**.** Let f be a bounded measurable function on a set of finite measure E. Assume g is bounded and $f = g$ a.e. on E.

Proof. Let $N = \{x \in E : f(x) \neq g(x)\}$. Note that $m(N) = 0$. Now, observe that

$$\int_E = \int_{E\setminus N} f - \int_N f = \int_{E\setminus N} g(x)dx + \int_N g(x)dx = \int_E gdx$$

since an integral over a set of measure zero is zero. □

Problem (13)**.** Show that the Bounded Convergence Theorem does not hold if $m(E) < \infty$ but we drop the assumption that the sequence $\{|f_n|\}$ is uniformly bounded on E.

Proof. Let

$$f_n(x) = \begin{cases} 2n^2x & \text{if } x \in \left[0, \frac{1}{2n}\right) \\ -2n^2x + 2n & x \in \left[\frac{1}{2n}, \frac{1}{n}\right) \\ 0 & \text{if } x \in \left[\frac{1}{n}, 1\right) \end{cases}$$

More simply, this is a function with a peak of width $\frac{1}{n}$ and height n at $\frac{1}{2n}$. For example, consider the following graphs generated by Wolfram Alpha.

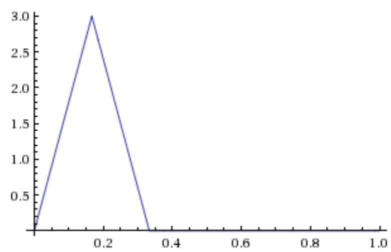

Figure 1: $n = 3$

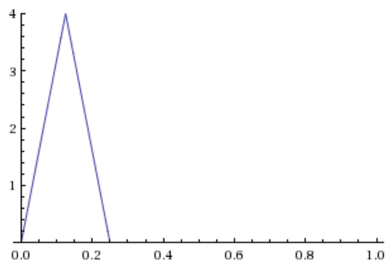

Figure 2: $n = 4$

Computing the intergral by area of triangles, we see that $\int_0^1 f_n = \frac{1}{2}\frac{1}{n}n = \frac{1}{2}$. Note, however, that

$$\lim_{n\to\infty} f_n(x) = 0$$

for all $x \in [0, 1]$. So, $\int_0^1 f_n = \frac{1}{2}$, but $\int_0^1 f = \int_0^1 0 = 0$. □

Problem (14)**.** Show that Proposition 8 is a special case of the Bounded Convergence Theorem.

Theorem. (Bounded Convergence Theorem) Let $\{f_n\}$ be a sequence of measurable functions on a set of finite measure E. Suppose $\{f_n\}$ is uniformly pointwise bounded on E, that is, there is a number $M \geq 0$ for which

$$|f_n| \leq M \text{ on } E \text{ for all } n.$$

Then,

$$\text{If } \{f_n\} \to f \text{ pointwise on } E\text{, then } \lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. Assume that $f_n(x) \to f(x)$ uniformly. Say that $|f_N(x)| < B$ for all x. Uniform convergence states that $|f_n(x) - f(x)| < \epsilon$ for all $n > N$. Thus, $|f_N(x) - f(x)| < \epsilon$, and thus

$$\epsilon < f(x) - f_N(x) < \epsilon$$

$$f_N(x) - \epsilon < f(x) < f_N(x) + \epsilon$$

and thus $|f(x)| < B + \epsilon$. Now, observe

$$|f_n - f + f - f_N| < 2\epsilon$$

by the triangle inequality, and thus

$$-2\epsilon < f_n(x) - f_N(x) < 2\epsilon$$

for all $n > N$ and thus $|f_n| \leq B + 2\epsilon$ and thus the Bounded Convergence Theorem applies. □

Problem (15)**.** Skip

Proof. Check continuity theorems of measure. □

Problem (16)**.** Let f be a nonnegative bounded measurable function on a set of finite measure E. Assume $\int_E f = 0$. Show that $f = 0$ a.e. on E.

Proof. It suffices to show that $m(\{x \in E : f(x) > \frac{1}{n}\}) = 0$ for all $n \in \mathbb{N}$, because $m(\{x : f(x) > 0\}) = \bigcup_{\frac{1}{n}>0}\{f(x) > \frac{1}{n}\}$.

Consider

$$\frac{1}{n}\chi_{\{x\in E:f(x)>\frac{1}{n}\}} \leq f(x)\chi_{\{x\in E:f(x)>\frac{1}{n}\}}$$

$$\int_E \frac{1}{n}\chi_{\{x\in E:f(x)>\frac{1}{n}\}} \leq \int_E f(x)\chi_{\{x\in E:f(x)>\frac{1}{n}\}} = \int_{\{x\in E:f(x)>\frac{1}{n}\}} f(x)dx$$

because

$$\int_E f(x) \geq \int_{E\setminus A} f$$

□

Problem (17)**.** Let E be a set of measure zero and define $f = \infty$ on E. Show that $\int_E f = 0$.

Proof. (Alexander's proof)
Define a function g on $\mathbb{R}$ by

$$g = \begin{cases} f & \text{if } x \in E \\ 0 & \text{if } x \in E^C \end{cases}$$

Since g is 0 a.e. on $\mathbb{R}$, by Proposition 9, $\int_{\mathbb{R}} g = 0$. Then,

$$0 = \int_{\mathbb{R}} g = \int_E g + \int_{E^C} g = \int_E f + \int_{E^C} 0 = \int_E f + 0 = \int_E f.$$

□

Proof. (Gatto's Proof) Note that f vanishes outside a set of finite measure. Thus,

$$\int_E f = \sup_{0\leq h\leq f} \int_E h$$

where h is bounded on E and vanishes outside of a set of finite measure. Note that since h is bounded, then $0 \leq h < M$ for some M. Thus,

$$\int_E h \leq \int_E M = M \cdot m(E) = 0.$$

□

Problem (18)**.** Show that the integral of a bounded measurable function of finite support is properly defined.

Proof. (Alexander's proof)
We want to show that, even if $m(E) = \infty$, defining the integral over E by

$$\int_E f = \int_{E_0} f$$

where E_0 has finite measure and $f = 0$ on $E \setminus E_0$, then this integral is properly defined.

Note that by Proposition 9, $\int_{E \setminus E_0} f = 0$. Thus,

$$\int_E f = \int_{E \setminus E_0} f + \int_{E_0} f = 0 + \int_{E_0} f = \int_{E_0} f.$$

□

Problem (19)**.** For a number α, define

$$f(x) = \begin{cases} x^\alpha & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Compute $\int_0^1 f$.

Proof. (Dr Gatto)
Begin with the case $\alpha < -1$. Apply the monotone convergence theorem. Define:

$$f_n = \begin{cases} x^\alpha, \text{ if } \frac{1}{n} \leq x \leq 1 \\ 0, \text{if } 0 \leq x \leq \frac{1}{n} \end{cases}$$

Note that $\int x^\alpha = \lim \int f_n = \lim \frac{x^{\alpha+1}}{\alpha+1}\Big|_{\frac{1}{n}}^{1} \to \infty$

If $\alpha = -1$, then

$$\int_0^1 x^\alpha = \lim_{n\to\infty} \int_{1/n}^1 f_n = \lim_{n\to\infty} \ln n \Big|_{1/n}^1 \to \infty$$

If $-1 < \alpha$, then

$$\int_0^1 x^\alpha = \lim_{n\to\infty} \int_{1/n}^1 x^\alpha = \lim_{n\to\infty} \frac{x^{\alpha+1}}{\alpha+1}\Big|_{1/n}^1 = \frac{1}{\alpha+1} > 0$$

□

Problem (20)**.** Let $\{f_n\}$ be a sequence of non-negative measurable functions that converges to f pointwise on E. Let $M \geq 0$ such that $\int_E f_n \leq M$ for all n. Show that $\int_E f \leq M$. Verify that this property is equivalent to the statement of Fatou's lemma.

Proof. (Alexander's proof)
By the definition of pointwise convergence, $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$. Then, we know $\int \lim_{n\to\infty} f_n(x) = \int f(x)$. Since this sequence of integrable functions converges pointwise everywhere on E to f,

$$\int_E f(x) = \int_E \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int_E f_n(x) \leq M.$$

Now we want to show that this property is equivalent to the statement of Fatou's lemma. Dr. Gatto says not to worry about this direction. □

Proof. (Dr. Gatto)
Apply Fatou's Lemma and the Monotone Convergence Theorem to see that

$$\int_E f = \int_E \lim_{n\to\infty} f_n \le \liminf_{n\to\infty} \int_E f_n \le M.$$

Conversely, to get Fatou... skip this portion. □

Problem (21)**.** Let the function f be nonnegative and integrable over E and $\epsilon > 0$. Show there is a simple function η on E that has finite support, $0 \le \eta \le f$ on E and $\int_E |f - \eta| < \epsilon$. If E is a closed, bounded interval show there is a step function h on E that has finite support and $\int_E |f - h| < \epsilon$.

Proof. (WARNING: WHAT FOLLOWS IS A MESS. I RECOMMEND AGAINST READING IT. DR GATTO SPENT AN HOUR ON THIS PROOF AND EVERYTHING WAS HORRIBLE)
Let $h_n = \min\{n, f(x)\}$. Then, the h_n are increasing, and since f is finite almost everywhere, $\lim_{n\to\infty} h_n = f(x)$ a.e. By monotone convergence,

$$\lim_{n\to\infty} \int h_n = \int f$$

which is equivalent to saying

$$\lim_{n\to\infty} \int f - h_n = 0$$

Thus, given $\frac{\epsilon}{2}$, there is an N_0 such that $\int f - h_{N_0} < \frac{\epsilon}{2}$. Since h_n is bounded and vanishes outside E, there is a sequence a sequence ϕ_ϵ of simple functions, where $0 \le \phi_\epsilon \le h_n$, such that $|h_n(x) - \phi_\epsilon(x)| < \frac{\epsilon}{2m(E)}$.

Let $\eta = \phi_\epsilon$. Then,

$$|f - \eta| = |f - h_{N_0} + h_{N_0} - \eta| \le |f - h_{N_0}| + |h_{N_0} - \eta|$$

and integrating we see that

$$\int |f - \eta| = \int |f - h_{N_0} + h_{N_0} - \eta| \le \int |f - h_{N_0}| + \int |h_{N_0} - \eta| \le \frac{\epsilon}{2} + \frac{\epsilon}{2m(E)} m(E) = \epsilon.$$

Now, $I = E$, where E is a closed, bounded interval. Then, there is a step function g such that $g = \eta$ except on $I \setminus F$, where $m(I \setminus F) < \frac{\epsilon}{2L}$. Now, since $\eta \le N_0$ and $g \le N_0$,

$$\begin{aligned}
\int_E |f - g| &\le \int_E |f - \eta| + \int_E |\eta - g| \le \frac{\epsilon}{2} \\
&\le \int_E |f - \eta| + \int_{I\setminus F} |\eta - g| + \int_F |\eta - g| \\
&= \int_E |f - \eta| + \int_{I\setminus F} |\eta - g| \\
&\le \epsilon
\end{aligned}$$

Let $\eta = \sum a_i \chi_{E_i}$, where $hj - \chi_{E_i}$ approximates each characteristic, but $h \le 1$, except $I \setminus F$, $m(I \setminus F) \le \frac{\epsilon}{3\ell}$.

We will approximate everything by doing the following: $L = \sum_{i=1}^{\ell} |a_i|$, so $a_j h_j = a_j \chi_{E_j}$. So, $\sum a_j h_j \le L$ and $\sum a_j \chi_{E_j} \le L$.

Everything in this writeup is wrong. Get notes from Christina. □

Problem (22)**.** Let $\{f_n\}$ be a sequence of non-negative measurable functions on $\mathbb{R}$ that converges pointwise on $\mathbb{R}$ to f and f be integrable over $\mathbb{R}$. Show that if

$$\int_{\mathbb{R}} f = \lim_{n\to\infty} \int_{\mathbb{R}} f_n, \text{ then } \int_E f = \lim_{n\to\infty} \int_E f_n \text{ for any measurable set } E$$

Proof. Since f_n are nonnegative, we apply Fatou. We know that

$$\int_E f \leq \liminf \int_E f_n.$$

The same is true for the complement:

$$\begin{aligned}\int_{\mathbb{R}} f - \int_E f = \int_{\mathbb{R}\setminus E} f \\ &\leq \liminf \int_{\mathbb{R}\setminus E} f_n \\ &\leq \liminf \left(\int_{\mathbb{R}} f_n - \int_E f_n \right) \\ &\leq \int_{\mathbb{R}} f - \limsup \int_E f_n\end{aligned}$$

Therefore,

$$\limsup_{n\to\infty} \int_E f_n \leq \int_E f$$

Combining all of this information, we observe that

$$\int_E f \leq \liminf \int_E f_n \leq \limsup_{n\to\infty} \int_E f_n \leq \int_E f$$

and thus

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

□

Problem (23)**.** Let $\{a_n\}$ be a sequence of non-negative real numbers. Define the functions f on $E = [1, \infty)$ by setting $f(x) = a_n$ if $n \leq x < n+1$. Show that $\int_E f = \sum_{n=1}^{\infty} a_n$.

Proof. In order to apply the Monotone Convergence Theorem, we construct the following sequence of partial sums:

$$S_j = \begin{cases} f(x), & \text{if } 1 \leq x < j \\ 0, & \text{if } x \geq j \end{cases}$$

Then, S_j is an increasing sequence of functions that converges pointwise to f. Therefore, we can apply the Monotone Convergence Theorem. Also note that

$$\int_E S_j = \int_{[1,j)} S_j + \int_{[j,\infty)} S_J = \int_{[1,j)} S_j = \sum_{n=1}^{j} a_n \cdot m\left([n, n+1)\right) = \sum_{n=1}^{j} a_n$$

Combining all of this information together, we see that

$$\int_E f = \lim_{j\to\infty} \int_E S_j = \lim_{j\to\infty} \sum_{n=1}^{j} a_n = \sum_{n=1}^{\infty} a_n.$$

□

Problem (24)**.** Let f be a non-negative measurable function on E.

1. Show there is an increasing sequence $\{\phi_n\}$ of non-negative simple functions on E, each of finite support, which converges pointwise on E to f.
2. Show that $\int_E f = \sup\left\{\int_E \phi : \phi \text{ simple, of finite support, and } 0 \le \phi \le f \text{ on } E\right\}$.

Proof. 1. Define

$$E_{j,k} = \left\{x : \frac{1}{2^k}(j-1) \le f(x) < \frac{1}{2^k}j\right\}, \text{ where } 1 \le j \le 4^k, \text{ and}$$
$$E_k^* = \left\{x : f(x) > 2^k\right\}$$

Use this to construct a sequence ϕ_k, where

$$\phi_k = \sum_{j=1}^{4^k} \frac{1}{2^k}(j-1)\chi_{E_{j,k}} + 2^k \chi_{E_k^*}$$

This is an increasing sequence of non-negative simple functions on E which converges pointwise to f. To ensure finite support, take

$$\psi_k = \chi_{[-k,k]} \cdot \phi_k.$$

2. Since $\{\phi_k\}$ are increasing, we will apply the Monotone Convergence Theorem:

$$\int_E f = \lim_{k\to\infty} \int_E \phi_k \le \sup_{\phi \le f} \int_E \phi$$

Furthermore, since $f \ge \phi \ge 0$,

$$\int_E f \ge \sup_{\phi \le f} \int_E \phi$$

and therefore

$$\int_E f = \sup_{\phi \le f} \int_E \phi.$$

□

Problem (25)**.** Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E that converges pointwise on E to f. Suppose $f_n \le f$ on E for each n. Show that

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

Proof. (Liz's proof)
By Fatou's lemma, $\int_E f \le \liminf_{n\to\infty} \int_E f_n$.

On the other hand, since $f_n \le f$ for each n, then $\limsup_{n\to\infty} \int_E f_n \le \int_E f$. Then, $\int_E f = \lim_{n\to\infty} \int_E f_n$. □

Proof. (Gatto's proof) Take $g_k(x) = \inf_{n \geq k} \{f_n(x)\}$, $\lim g_k(x) = f(x)$. Then, by Fatou,

$$\int_E f \leq \liminf \int f_k$$

Next, consider $f - f_n \to 0$ a.e. Also, $f_n \leq f$ and thus $f - f_n$ is positive, so we can use Fatou again. Then,

$$\int 0 \leq \lim_{n\to\infty} \inf \int (f - f_n)$$

and thus

$$0 \leq \int f - \lim_{n\to\infty} \sup \int f_n$$

implying that

$$\int f \geq \lim_{n\to\infty} \sup \int f_n.$$

□

Problem (26)**.** Show that the Monotone Convergence Theorem may not hold for decreasing sequences of functions.

Proof. (****ON TEST PROBABLY****)
Consider a set $E = [1, \infty)$, and define a sequence $f_n = \chi_{[n,\infty)}$. Observe that $\int_E f_n = \int_E \chi_{[n,\infty)} = \infty$. On the other hand, $\int_E f = \int_E \lim_{n\to\infty} \chi_{[n,\infty)} = \int_E 0 = 0$. □

Problem (27)**.** Prove the following generalization of Fatou's Lemma: If $\{f_n\}$ is a sequence of nonnegative measurable functions on E, then

$$\int_E \liminf f_n \leq \liminf \int_E f_n.$$

Proof. (Gatto's Proof)
Take $g_k(x) = \inf_{n\geq k}\{f_n\}$. Now,

$$\lim_{k\to\infty} g_k(x) = g(x) = \lim_{n\to\infty} \inf\{f_n(x)\}$$

so we can apply Fatou.

$$\int \lim g_k \leq \lim_{k\to\infty} \inf \int g_k \leq \liminf \int f_k$$

and thus

$$\int \liminf\{f_k\} \leq \liminf \int f_k.$$

□